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$$\frac{f(k)}{10^{k-1}} (1 + r + r^2 + \cdots).$$

This series converges, for $r$ is $< 1$; and hence our given series converges.

The case of "improper" combinations may be rapidly dealt with by the following device. Write the digit $a$, $k$ times at the end of the given combination, where $a$ is any digit other than the first digit of the combination. This certainly gives us a "proper" combination (of $2k$ digits), so that, as already proved, the series formed from the harmonic series by omitting terms whose denominators contain this combination converges. But this series evidently contains all terms of the given series; this latter, then, converges too.

In like manner we may prove that *the series obtained by omitting terms containing all of a given set of combinations (some of which may be repeated) converges.* Here we may compare the series with that obtained by omitting terms containing the single combination formed from the given combinations by juxtaposition.

This supplies another proof of the proposition in my paper on "A Curious Convergent Series" in this MONTHLY for May, 1916, that proposition being the special case of the above in which each of the combinations consists of a single digit.

The proposition stated by Professor Moulton in his solution of Algebra 453 in this MONTHLY for Oct., 1916, refers, I suppose—the proof is not given in full—not to combinations of the kind here considered, but to selections, and without any reference to the juxtaposition of the digits of the selection in the omitted terms.

**239** (**Number Theory**) [**March, 1916; March, 1919**].  **Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.**

Give a general method for determining the solution in integers of the equation

$$x^r - 10xy - (n + 1) + y = 0,$$

where $r$ and $n$ are positive integers.

## II. SOLUTION BY FRANK IRWIN, University of California.

If $x = 0$, $y = n + 1$; noting this solution, we shall suppose in what follows $x \neq 0$.

The degree in $x$ of the given equation may be reduced by the following device. $y - (n + 1)$ must be divisible by $x$, say $y - (n + 1) = y_1 x$. Then our equation reduces to

$$x^{r-1} - 10xy_1 - 10(n + 1) + y_1 = 0.$$

Putting $y_1 - 10(n + 1) = y_2 x$, $y_2 - 10^2(n + 1) = y_3 x$, etc., we finally reach the equation

$$1 - 10xy_r - 10^r(n + 1) + y_r = 0, \tag{1}$$

where $y$ and $y_r$ are connected, it will be found, by the relation

$$y = (n + 1)[1 + 10x + 10^2 x^2 + \cdots + 10^{r-1} x^{r-1}] + y_r x^r,$$

or

$$y = (n + 1) \frac{10^r x^r - 1}{10x - 1} + y_r x^r. \tag{2}$$

Rewriting (1) as

$$(10x - 1)y_r = 1 - 10^r(n + 1), \tag{3}$$

we see that $x$ and $y_r$ must have opposite signs.

(*i*) Let $y_r$ be negative, $y_r = - y_r'$. Then (3) becomes $(10x - 1)y_r' = 10^r(n + 1) - 1$. Here the number on the right ends with the digit 9, as does also the first factor on the left. Hence we see that $y_r'$ must end with 1. We get, then, a solution by taking for $y_r'$ any factor, ending with 1, of the right side, and determining $x$ and $y$ from (3) and (2).

(*ii*) Let $x$ be negative, $x = - x'$. Then (3) becomes

$$(10x' + 1)y_r = 10^r(n + 1) - 1.$$

We take $y_r$ equal to any factor of the right side that ends with 9.

This gives us all solutions.

Suppose that $10^r(n + 1) - 1$ can be factored as $a \cdot b$, where $a$ ends with 1, $b$ with 9. Then

we may take either $y_r = -a$, whence $x = (b+1)/10$; or $y_r = b$, $x = (-a+1)/10$. Thus solutions go in pairs, the $x$ of either solution being obtained by adding 1 to the $y_r$ of the other and dividing by 10. There are, then, always an even number of solutions, and always at least two, viz., those where $y_r = -1$, $x = 10^{r-1}(n+1)$; $y_r = 10^r(n+1) - 1$, $x = 0$.

Finally it may be noted that the more general equation

$$ax^r - bxy + y - c = 0$$

may be solved by similar methods. Here we should have to pick out such divisors of $b^r c - a$ as are $\equiv \pm 1$ (mod. $b$). It may be interesting to apply the method to an example. Let us solve

$$x^4 - 10xy - 22 + y = 0.$$

We have to solve

$$(10x - 1)y_4 = -219999 = -3 \times 13 \times 5641$$

(5641 prime). The values of $y_4$ with the corresponding solutions of our equation are:

$$y_4 = -5641, \quad x = 4, \quad y = 6; \quad y_4 = -1, \quad x = 22000 \text{ ($y$ a very large number)};$$

$$y_4 = 39, \quad x = -564, \quad y = -17937434; \quad y_4 = 219999, \quad x = 0, \quad y = 22.$$

**247 (Number Theory) [June, 1916]. Proposed by NORMAN ANNING, Chilliwack, B. C.**

To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangs-gründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

### SOLUTION BY FRANK IRWIN, University of California.

The orthocenter, $O$, may be taken as the required point. Let $ABC$ be the triangle with $a = 60$, $b = 52$, $c = 56$; and let the feet of the perpendiculars from $A, B, C$ on the opposite sides be $D, E, F$, respectively. Then the various lines in the figure, calculated as indicated below, are: $BD = 168/5$, $DC = 132/5$, $CE = 396/13$, $EA = 280/13$, $AF = 20$, $FB = 36$; $AO = 25$, $BO = 39$, $CO = 33$. Finally, area $BOC = 594$, area $COA = 330$, and area $AOB = 420$; so that the sides and areas of these three triangles are integral, as asserted.

The explanation of these facts depends on the following proposition: If the sides and area of the triangle $ABC$ are rational, the same is true of the triangles $BOC, COA, AOB$. (Then by multiplying the dimensions of the figure by a suitable integer everything can be made integral.) For the three altitudes are rational, as also the radius $r$ of the inscribed circle (since $rs = $ area). Thus $\tan A/2$ is rational, and so, then, are $\cos^2 A/2$ and $\cos A$. Therefore, $AF = b \cos A$ is rational, and similarly, $FB, BD$, etc. Then the triangle $AOF$ is rational (that is, has rational sides), since one of its sides, $AF$, is rational, and it is similar to the rational triangle $ABD$.
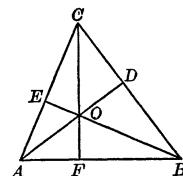
**2678 [February, 1918]. Problem proposed by C. F. GUMMER, Queen's University, Canada.**

Find necessary and sufficient conditions that the roots of the equation $x^{n+1} + a_1 x^n + a_2 x^{n-1} + \cdots + a_{n+1} = 0$ may be all real and separated by the roots of $x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n = 0$.

### SOLUTION BY THE PROPOSER.

Consider the equations

(1) $\quad f(x) \equiv x^{n+1} + a_1 x^n + \cdots \qquad = 0,$

(2) $\quad g(x) \equiv x^n + b_1 x^{n-1} + \cdots \qquad = 0,$

(3) $\quad R_1(x) \equiv c_0 x^{n-p} + c_1 x^{n-p-1} + \cdots \qquad = 0,$

(4) $\quad R_2(x) \equiv d_0 x^{n-p-q} + d_1 x^{n-p-q-1} + \cdots \; = 0,$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where $R_1(x)$ is the remainder with sign changed on dividing $f(x)$ by $g(x)$, $R_2(x)$ has the same relation to $g(x)$ and $R_1(x)$, etc.